

Volumetric Sensible-Enthalpy Formulation for Coupled Transport in MHD Casson Nanofluid Flow: Revealing Energy Pathways beyond Wall Fluxes

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Abstract

This study presents a comprehensive analysis of coupled momentum, heat, and mass transfer in MHD Casson nanofluid flow over a stretching surface using a volumetric sensible-enthalpy formulation. Unlike conventional temperature-based models, the approach explicitly resolves internal energy generation, storage, and redistribution arising from viscous dissipation, Joule heating, buoyancy forces, and nanoparticle transport. The governing nonlinear equations for momentum, enthalpy, and concentration are transformed via similarity variables and solved numerically using a collocation method. Numerical investigations show that magnetic fields strongly suppress fluid motion while enhancing internal energy accumulation, reducing surface heat and mass transfer. Thermal and solutal buoyancy synergistically intensify momentum, heat, and species transport. Viscous dissipation primarily contributes to volumetric energy storage rather than wall heat flux, whereas Brownian motion significantly redistributes internal enthalpy with negligible effect on skin friction. Increasing the Casson parameter enhances convective enthalpy transport by weakening yield-stress resistance. The results demonstrate that wall fluxes alone may not reliably indicate internal energy dynamics, highlighting the enthalpy-based framework as a unified tool for capturing coupled transport pathways in non-Newtonian MHD nanofluids.

Keywords: Boundary layer; viscous dissipation; Brownian motion; Magnetic field; stretching sheet; nonlinear, ODE

1. Introduction

Nanofluids, formed by dispersing nanoscale solid particles into conventional base fluids, have emerged as advanced heat-transfer media with enhanced thermal and transport characteristics compared to pure liquids [1]. Among these, Al_2O_3 ethylene glycol (EG) suspensions have received significant attention due to their superior thermal conductivity, stability, and compatibility with industrial cooling and energy systems [2]. The interaction between momentum, thermal, and solutal diffusion in such nanofluid flows plays a crucial role in determining heat and mass transport efficiency, especially in systems where magnetic fields, viscous dissipation, and Brownian motion are simultaneously active [3,4]. In electrically conducting nanofluids, the application of a transverse magnetic field induces a Lorentz force that resists motion, thereby influencing both hydrodynamic and thermal boundary layers. The study of magnetohydrodynamic (MHD) nanofluid flow has therefore become fundamental to modern technologies such as micro-electromechanical systems (MEMS), magnetically

controlled cooling, thermal energy storage, and biomedical coating flows. Furthermore, Casson rheology provides a robust model for non-Newtonian fluid structures relevant to polymeric suspensions, paints, and biofluids, capturing yield-stress effects that strongly couple with viscous and thermal fields [5, 6].

Aluminium oxide (Al_2O_3) nanoparticles are chemically stable, non-toxic, and cost-effective, which has made them among the most extensively studied nanomaterials. These nanoparticles possess high thermal conductivity and can be uniformly dispersed in a variety of base fluids, with ethylene glycol (EG) being particularly effective. Ethylene glycol itself is widely used as a base fluid due to its low volatility, moderate viscosity, and good compatibility with metal oxide nanoparticles. Its properties make it an ideal medium for suspending nanoparticles and it is commonly employed in automotive and electronic cooling applications. Nanofluids composed of Al_2O_3 ethylene glycol have demonstrated notable enhancements in thermal conductivity, specific heat capacity, and convective heat transfer performance, making them promising candidates for advanced cooling systems. For instance [7], conducted a numerical investigation on heat transfer enhancement using a water/ethylene glycol mixture containing $Al_2O_3 - TiO_2$ nanoparticles. [8], analyzed the thermal performance of Al_2O_3 -water nanofluid flow in composite microchannels. Similarly, [9] studied the heat transfer characteristics of an ethylene glycol-based Casson nanofluid around a horizontal circular cylinder under MHD effects, reporting that CuO nanofluid exhibited the largest thermal boundary layer thickness and the lowest Nusselt number. A method-of-moments analysis on $EG - Al_2O_3$ nanofluid flow through an expanding/contracting rectangular channel was conducted by [10].

Their results reveal that channel contraction intensifies velocity gradients and heat transfer, while expansion weakens the overall transport characteristics. The general trend from their study is that geometric deformation directly regulates nanofluid momentum and thermal diffusion. [11], developed and evaluated six supervised machine-learning models to predict the thermal conductivity of Al_2O_3 -ethylene glycol nanofluids. Their findings show that data-driven models can accurately estimate thermal conductivity based on nanoparticle concentration and fluid properties, with tree-based methods achieving the highest predictive accuracy. The general conclusion from their work is that machine-learning techniques offer a reliable alternative to empirical correlations for nanofluid thermal characterization. [12], performed detailed measurements of the thermal conductivity and viscosity of Al_2O_3 -ethylene glycol nanofluids across various temperatures and particle loadings. Their results show that adding alumina nanoparticles significantly increases thermal conductivity, with the enhancement becoming more pronounced at higher temperatures due to intensified Brownian motion. They also reported a substantial rise in viscosity with particle concentration, indicating a trade-off between improved heat transfer and increased flow resistance. The general implication of their study is that while EG-based alumina nanofluids offer superior thermal performance, their rheological penalties must be carefully considered in system design and pumping-power calculation. [13], conducted both experimental and numerical investigations into turbulent forced-convection heat transfer of Al_2O_3 -ethylene glycol nanofluids in a flat tube. They found that introducing Al_2O_3 nanoparticles into the ethylene glycol base fluid significantly increases the convective heat transfer coefficient, particularly at

higher Reynolds numbers, while numerical simulation validated the experimental trends across various nanoparticle volume fractions.

The general implication of their work is that Al_2O_3 –ethylene glycol nanofluids offer a viable route for heat-transfer augmentation in turbulent flow regimes, though accompanying increases in viscosity and potential frictional losses must be considered for practical implementation. [14], examined the turbulent forced convective flow of an Al_2O_3 –ethylene glycol nanofluid in a circular tube under constant wall-temperature conditions, with the added focus on entropy generation minimization through genetic-algorithm (GA) and direct optimization techniques. Their results show that convective heat transfer improves with increasing Reynolds number and nanoparticle volume concentration, but also that entropy generation rises with Reynolds number and smaller particle sizes thus revealing a trade-off between heat transfer enhancement and irreversibility. The general implication of their work is that optimal nanoparticle size and volume fraction exist which minimize entropy generation while maintaining improved thermal performance in EG-based alumina nanofluids. [15], carried out a numerical investigation of a hybrid nanofluid consisting of Al_2O_3 – Cu nanoparticles dispersed in ethylene glycol, modeled through a non-Newtonian Maxwell fluid formulation under MHD conditions over a stretching inclined sheet with suction in a porous medium. Their results demonstrate that the fluid velocity increases with the Maxwell fluid parameter, suction parameter and inclination angle, while both increasing the magnetic field strength and the porous medium parameter tend to reduce velocity but increase temperature. They found that the Nusselt number (heat-transfer rate) responds sensitively to suction and nanoparticle volume fractions, suggesting that the combination of MHD, non-Newtonian rheology and hybrid nanoparticle dispersion strongly influences thermal efficiency. The general implication of their study is that hybrid Al_2O_3 – Cu/EG nanofluids, when used under MHD and non-Newtonian frameworks, can achieve enhanced heat-transfer behaviour, but the interplay between magnetic, porous and rheological parameters must be carefully managed. [16], investigated the heat-transfer performance of Al_2O_3 , SiO_2 and ZnO nanoparticle suspensions in ethylene glycol–water ($EG - H_2O$) base fluids within a rectangular microchannel under laminar flow conditions. Their results show that increasing nanoparticle volume fraction and higher EG content in the base fluid led to enhanced convective heat transfer coefficients; for example, a 40% ($EG - H_2O$) mixture produced the highest increase with ZnO at 5% volume fraction. They also observed that pressure drop and pumping power rise significantly with EG ratio and nanoparticle loading, with ZnO showing the largest penalty.

The general implication is that while EG –based nanofluids with Al_2O_3 (and other oxides) can achieve considerable heat-transfer improvement in microchannel applications, system design must carefully balance enhancement against increased hydrodynamic losses. [17], carried out a numerical investigation of magnetohydrodynamic (MHD) boundary-layer flow of Al_2O_3 –ethylene glycol nanofluids (in an ethylene-glycol/water base) past a melting surface, introducing an effective Prandtl number model to account for nanoparticle effects. Their results demonstrate that inclusion of the effective Prandtl number leads to a thicker momentum and thermal boundary layer compared with conventional values, and that the transverse magnetic field suppresses fluid motion while enhancing thermal boundary-layer thickness. The general implication from their study is that in electrically

conductive nanofluids, both magnetic field and nanoparticle-modified transport properties (via effective Prandtl number) must be incorporated to accurately model melting heat-transfer processes. Similarly, [18] investigated the combined effects of Brownian motion, thermophoresis and magnetic field on a hybrid nanofluid flowing over a stretched wedge surface. Their results show that increasing the wedge angle enhances the local Nusselt number, whereas stronger magnetic field strength thickens the thermal boundary layer and reduces heat-transfer efficiency. The general implication of their work is that in hybrid nanofluid systems, the interaction of nanoparticle-diffusive mechanisms (Brownian, thermophoretic) with magnetic suppression and geometric parameters must be accounted for in heat-transfer models. [19], investigated the effects of Brownian motion and thermophoresis on nanofluid flow in channels using a two-phase model. Their results show that nanoparticle migration significantly alters both temperature and concentration fields, demonstrating that diffusive particle transport strongly enhances coupled heat-mass transfer. The general implication of their findings is that Brownian and thermophoretic forces must be included to accurately capture nanofluid behavior in confined geometries.

From the literature reviewed, it is evident that conventional nanofluid models predominantly employ temperature-based formulations of the energy equation and the thermal buoyancy. While these approaches have yielded valuable insights, they often overlook the fundamental volumetric enthalpy coupling that intrinsically links fluid density, heat capacity, and the temperature field. This oversight becomes particularly significant when considering the complex interplay between multiple transport mechanisms in Casson nanofluids under magnetic fields. To address this limitation, the present work introduces a volumetric sensible enthalpy formulation: $\hat{h} = \rho_{nf} c_{p,nf} (T - T_\infty)$, which directly embeds the nanofluids thermo-physical properties into the thermal buoyancy and the governing energy transport description. This formulation naturally captures the dynamic feedback between heat generation, viscous dissipation, Joule heating, and nanoparticle diffusion (Brownian motion), providing a more physically complete framework for multi-parameter volumetric sensible enthalpy transport analysis.

The primary objectives of this study are to analyze the influence of magnetic field intensity, viscous dissipation, buoyancy force, interacting transport parameters (Brownian motion, Schmidt number) and other thermophysical properties on the velocity, enthalpy, and concentration distributions, and quantify how these coupled effects alter skin-friction, Nusselt number, and Sherwood number characteristics within a volumetric enthalpy framework. While mathematically equivalent to temperature formulations for constant property cases, the volumetric enthalpy approach provides a more fundamental thermodynamic framework that directly represents the actual energy content per unit volume, offers enhanced physical insight into coupled transport mechanisms, and establishes a foundation for extending to more complex scenarios involving property variations, phase change, or large temperature ranges.

Most existing studies on MHD Casson nanofluid boundary-layer flow formulate thermal transport using temperature as the primary variable, thereby restricting analysis to wall heat-flux behaviour and external thermal boundary-layer characteristics. In contrast, the present work introduces a volumetric sensible-enthalpy formulation in which the energy equation and the thermal buoyancy are recast in terms of internal energy density, coupled with enthalpy-dependent thermal conductivity and

nanoparticle transport effects. This framework enables direct examination of internal energy storage, redistribution, and generation mechanisms within the nanofluid, providing physical insight that cannot be obtained from temperature-based models. The formulation reveals how Lorentz-force damping, Brownian diffusion, viscous dissipation, and dual buoyancy forces collectively regulate volumetric energy pathways rather than merely surface heat flux. Consequently, the study establishes magnetic-enthalpy coupling as a new control mechanism for internal thermal management in electrically conducting Casson nanofluids, offering a fundamentally different perspective on energy transport beyond conventional wall-flux-oriented analyses.

2. Mathematical Formulation of the Model

Consider a steady, laminar, two-dimensional boundary-layer flow of an electrically conducting Casson nanofluid consisting of Al_2O_3 nanoparticles dispersed in ethylene glycol (EG). The flow is induced by a continuously stretching sheet with a linear velocity $U_w = ax$, where $a > 0$ is the constant stretching rate. A uniform transverse magnetic field of strength B_0 is applied perpendicular to the surface. Nanoparticle transport is governed by Brownian motion, while thermal energy transport is described using a volumetric sensible enthalpy formulation defined by $\hat{h}(x, y) = \rho_{nf} c_{p,nf} (T - T_\infty)$, replacing the classical temperature variable in the thermal buoyancy and energy equation. Let $u(x, y)$ and $v(x, y)$ denote the velocity components in the streamwise (x) and transverse (y) directions, respectively. The dependent variables are the velocity field, the volumetric sensible enthalpy $\hat{h}(x, y)$, and the nanoparticle concentration $C(x, y)$ as displayed in Figure 1.

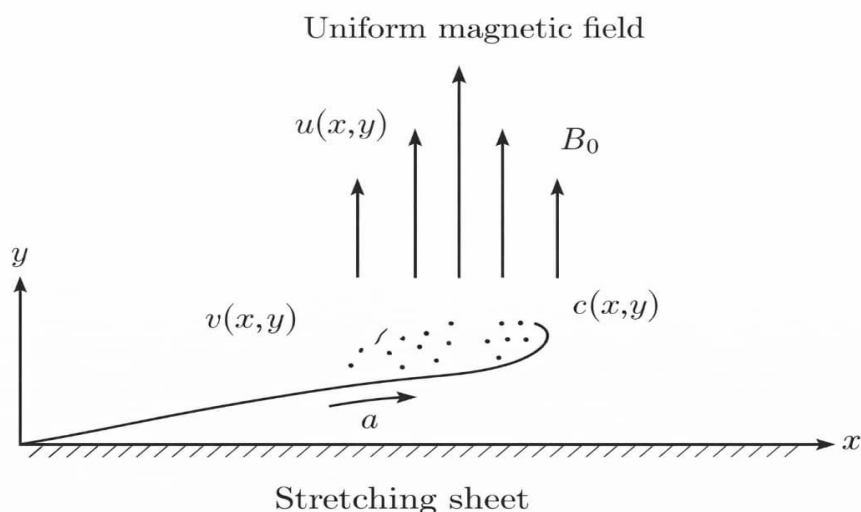


Figure 1. Schematic Diagram of the flow

The governing partial differential equations, satisfying boundary-layer assumptions are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\mu_{nf}}{\rho_{nf}} \left(1 + \frac{1}{\beta} \right) \frac{\partial^2 u}{\partial y^2} + g\beta_T \left(\frac{\hat{h}}{\rho_{nf} c_{p,nf}} \right) + g\beta_C (C - C_\infty) - \frac{\sigma_{nf} B_0^2 u}{\rho_{nf}} \quad (2)$$

$$\begin{aligned}
 u \frac{\partial \hat{h}}{\partial x} + v \frac{\partial \hat{h}}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{\kappa(\hat{h})}{\rho_{nf} c_{p,nf}} \frac{\partial \hat{h}}{\partial y} \right) + \tau D_B \frac{\partial C}{\partial y} \frac{1}{\rho_{nf} c_{p,nf}} \frac{\partial \hat{h}}{\partial y} + \frac{\mu_{nf}}{\rho_{nf} c_{p,nf}} \left(\frac{\partial u}{\partial y} \right)^2 \\
 &\quad + - \frac{\sigma_{nf} B_0^2 u^2}{\rho_{nf} c_{p,nf}} \quad (3) \\
 u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} &= D_B \frac{\partial^2 C}{\partial y^2} \quad (4)
 \end{aligned}$$

With the boundary conditions

$$\begin{aligned}
 u = ax, \quad v = 0, \quad -\kappa_f \frac{\partial \hat{h}}{\partial y} &= h_f (\hat{h}_f - \hat{h}), \quad C = C_w(x), \quad \text{at } y = 0 \\
 u \rightarrow 0, \quad \hat{h} \rightarrow \hat{h}_\infty, \quad C &\rightarrow C_\infty, \quad \text{at } y \rightarrow \infty \quad (5)
 \end{aligned}$$

Here u, v are the velocity components along x and y axis respectively, μ_{nf} and ρ_{nf} denote the dynamic viscosity and density of the nanofluid, ν is the kinematic viscosity, represents the variable thermal conductivity of the fluid as a function of enthalpy; β is the Casson fluid parameter, g denotes the gravitational acceleration; β_T and β_C are the thermal and solutal expansion coefficients, respectively. Furthermore, $\hat{h}$ is the dimensional volumetric enthalpy of the fluid (dimensional enthalpy), while $\hat{h}_\infty$ denotes the dimensional ambient (free stream) enthalpy far from the surface, which is taken as the reference enthalpy ($\hat{h}_\infty = 0$). The quantity $\hat{h}_w$ represents the dimensional wall (surface) enthalpy. The local temperature of the fluid, which is enthalpy-dependent, is denoted by T , while C represents the nanoparticle concentration. The symbols C_w and C_∞ denote the wall and free-stream concentrations, respectively. In addition, β_0 is the applied magnetic field strength, σ_{nf} denote the electrical conductivity of the nanofluid, $(\rho c_p)_{nf}$ is the effective heat capacity of the nanofluid, α_{nf} represent the thermal diffusivity of the nanofluid. The parameters D_B and D_T is Brownian diffusion and thermophoretic diffusion coefficient respectively. The quantity κ_f is the thermal conductivity of the base fluid, h denotes the dimensionless volumetric enthalpy corresponding to the fluid temperature, and $\hat{h}_f$ represent the volumetric enthalpy at fluid temperature, h_f is convective heat transfer coefficient of the base fluid, $U_w(x)$ is the stretching velocity along the plate, $C_w(x)$ is the concentration distribution along the plate, ν_{nf} and ν_{bf} denote the kinematic viscosities of the nanofluid and the base fluid, respectively.

The present model is formulated using the concept of volumetric sensible enthalpy of the nanofluid defined as

$$\hat{h} = \rho_{nf} c_{p,nf} (T - T_\infty), \quad (7)$$

from which the local temperature field is

$$T = T_\infty + \frac{\hat{h}}{\rho_{nf} c_{p,nf}}. \quad (8)$$

Where, ρ_{nf} is the effective density $c_{p,nf}$ the specific heat capacity at constant pressure, and T the local temperature expressed in terms of volumetric sensible enthalpy. This formulation establishes a direct correspondence between temperature and enthalpy,

allowing the temperature field to be entirely replaced by its enthalpy equivalent. Such a transformation simplifies the representation of internal volumetric effects, including viscous dissipation, Joule heating, magnetic field interactions, and Brownian motion of nanoparticles, within the boundary-layer framework. The incorporation of Brownian motion is essential for capturing nanoparticle-induced energy transport, which directly influences the local enthalpy and, consequently, the thermal behavior of the nanofluid. Following the approaches of [16, 20-22], the thermal conductivity of the flow is assumed to vary linearly with volumetric sensible enthalpy that is,

$$\kappa(\hat{h}) = \kappa^*(1 + \gamma h), \quad (9)$$

where κ^* represents the ambient thermal conductivity of the nanofluid, and h is the dimensionless enthalpy. In this study, the modified thermal conductivity variation parameter is defined as

$$\gamma = \delta \hat{h}_w, \quad (10)$$

such that δ is the thermal conductivity variation coefficient with units m^3/J , representing the rate of change of thermal conductivity with respect to the local volumetric enthalpy. Non-dimensionalization of the thermal diffusion coefficient, thermal diffusivity is defined as

$$\alpha_{nf} = \frac{\kappa^*}{\rho_{nf} c_{p,nf}}. \quad (11)$$

This parameter measures how rapidly heat diffuses through the nanofluid relative to its ability to store thermal energy, and it is the natural scaling quantity used when non-dimensionalizing thermal diffusivity in the enthalpy (energy) equation. The following properties for the nanofluid adopted [22-27] consist of effective density

$$(\rho_{nf}) = (1 - \phi)\rho_f + \phi\rho_p, \quad \text{dynamic viscosity } \frac{\mu_{nf}}{\mu_{nf} \nu_{bf}} = \frac{0.904e^{0.148\phi}}{1 - \phi + \phi \frac{\rho_{np}}{\rho_{bf}}}, \quad \text{specific heat:}$$

$$(\rho c_p)_{nf} = (1 - \phi)(\rho c_p)_f + \phi(\rho c_p)_p, \quad \text{thermal diffusivity } \alpha_{nf} = \frac{\kappa_{nf}}{\rho_{nf} c_{p,nf}}, \quad \text{and thermal}$$

$$\text{conductivity: } \frac{\kappa_{nf}}{\kappa_{bf}} = \frac{\kappa_{np} + 2\kappa_{bf} - 2\phi(\kappa_{bf} - \kappa_{np})}{\kappa_{np} + 2\kappa_{bf} + \phi(\kappa_{bf} - \kappa_{np})}.$$

Reducing the governing partial differential Equations (1)-(4) subject to the boundary conditions (5) and (6), the following similarity variables are introduced:

$$\eta = y \sqrt{\frac{a}{\nu_f}}, \quad \varphi(x, y) = \sqrt{a \nu_f} x f(\eta), \quad u = \frac{\partial \varphi}{\partial y}, \quad v = -\frac{\partial \varphi}{\partial x}, \quad (12)$$

$$\hat{h} = \hat{h}_w h(\eta) \quad (13)$$

$$C = C_\infty + (C_w - C_\infty)\phi(\eta). \quad (14)$$

The variable $h(\eta)$ which represents the dimensionless volumetric sensible enthalpy and satisfies the following boundary conditions is $h(\eta) = 0$ when $\hat{h}_\infty = 0$ (ambient condition, $T = T_\infty$), and $h(\eta) = 1$ when, $\hat{h} = \hat{h}_w$ (wall condition, $T = T_w$).

Substituting Equations (7)-(14) into Equations (1)-(4) reveals that Equation (1) is satisfied, while Equations (2)-(4), together with the boundary conditions (5)-(6), reduce to

$$\frac{\mu_{nf}}{v_f \rho_{nf}} \left(1 + \frac{1}{\beta} \right) f'''(\eta) + f(\eta) f''(\eta) - f'^2(\eta) + Gr_t + Gr_s - M f' = 0 \quad (15)$$

$$(1 + \gamma h) h''(\eta) + \gamma h'^2(\eta) + Pr_{nf} f(\eta) h'(\eta) + Nb \phi'(\eta) h'(\eta) + Br_h (f''(\eta))^2 + Pr_{nf} M Ec_h (f'(\eta))^2 = 0 \quad (16)$$

$$\phi'' + Sc f \phi' = 0. \quad (17)$$

Where prime denotes the differentiation with respect to η , $Gr_t = \frac{g \beta_T \hat{h}}{\rho_{nf} c_{p,nf} \alpha x^2}$, $Gr_s = \frac{g \beta_C (C_w - C_\infty)}{\alpha x^2}$, $Br_h = \frac{\mu_{nf} a^2 x^2}{\alpha_{nf} \rho_{nf} c_{p,nf} (\hat{h}_w)} \equiv \frac{\mu_{nf} U^2}{\kappa_{nf} (\hat{h}_w)}$, $Nb = \frac{\tau_{DB} (C_w - C_\infty)}{\kappa_{nf}}$, $Ec_h = \frac{a^2 x^2}{\rho_{nf} c_{p,nf} \hat{h}_w}$, $M = \frac{\sigma_{nf} B_0^2}{\rho_{nf} a}$, $Sc = \frac{v_f}{D_B}$ represents the enthalpy based thermal Grashof number, solutal Grashof number, volumetric Brinkman number, Brownian motion parameter, Eckert number, magnetic field parameter, and Schmidt number respectively. While $v_{nf} = \frac{\mu_{nf}}{\rho_{nf}}$ and $v_f = \frac{\mu_f}{\rho_f}$ denotes the kinematic viscosities of the nanofluid and the base fluid, subject to the dimensionless boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad h'(0) = -Bi(h(0) - \gamma), \quad \phi(0) = 1 \quad \text{at} \quad \eta = 0 \quad (18)$$

$$f'(\eta) \rightarrow 0, \quad h(\eta) \rightarrow 0, \quad \phi(\eta) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \quad (19)$$

Employing these transformations, the coupled nonlinear ordinary differential Equations (15)–(17), with the dimensionless boundary conditions (18)–(19), are reduced to a system of linear ordinary differential equations. Let, $y_1(\eta) = f(\eta)$, $y_2(\eta) = f'(\eta)$, $y_3(\eta) = f''(\eta)$, $y_4(\eta) = h(\eta)$, $y_5(\eta) = h'(\eta)$, $y_6(\eta) = \phi(\eta)$, $y_7(\eta) = \phi'(\eta)$ such that

$$A = \frac{\mu_{nf}}{v_f \rho_{nf}} \left(1 + \frac{1}{\beta} \right) \quad (20)$$

$$\text{The corresponding first-order system associated with Equations (15)–(17) is given as} \\ y_1' = y_2 \quad (21)$$

$$y_2' = y_3 \quad (22)$$

$$y_3' = \frac{-y_1 y_3 + y_2^2 - Gr_t - Gr_s + M y_2}{A} \quad (23)$$

$$y_4' = y_5 \quad (24)$$

$$y_5' = \frac{-\gamma y_5^2 - Pr_{nf} y_1 y_5 - Nby_7 y_5 - Br_h y_3^2 - Pr_{nf} M Ec_h y_2^2}{1 + \gamma y_4} \quad (25)$$

$$y_6' = y_7 \quad (26)$$

$$y_7' = -Sc y_1 y_7, \quad (27)$$

subject to the reformulated boundary conditions

$$y_1(0) = 0, y_2(0) = 1, y_5(0) = -Bi(y_4(0) - \gamma), y_6(0) = 1, \\ y_1(0) = 0, y_2(0) = 1, y_3(0) = s_1, y_4(0) = s_2, y_5(0) = -Bi(s_2 - \gamma), y_6(0) = 1, \\ y_7(0) = s_3 \quad (29)$$

with initial conditions,

$$y_2(\infty) = 0, y_4(\infty) = 0, y_6(\infty) = 0$$

where, $s_1 = f''(0)$, $s_2 = h(0)$, and $s_3 = \phi'(0)$ represent the unknown initial slopes of the velocity, enthalpy, and concentration profiles, respectively. These parameters are iteratively adjusted until the far-field boundary conditions $y_2(\infty) = 0$, $y_4(\infty) = 0$, and $y_6(\infty) = 0$ are satisfied within a prescribed tolerance. The coupled nonlinear Equation (19) is solved using the collocation method implemented in SciPy's solve_bvp routine. This approach combines collocation with domain discretization to produce accurate and stable solutions for two-point boundary-value problems. Nonlinear differential systems arising in transport phenomena are commonly treated using collocation-based boundary-value solvers, while alternative numerical frameworks such as block linear multistep schemes for initial-value formulations have also been proposed in recent computational studies [28], reflecting continuous development of numerical strategies for complex transport models.

Using the similarity transformations in Equations (12)–(14), the skin-friction coefficient, Nusselt number, Sherwood number, and wall enthalpy are evaluated in terms of the similarity functions, and the corresponding wall flux relations; shear stress, sensible enthalpy flux, and mass flux are given as

$$\tau_w = \mu_{nf} \left(\frac{\partial u}{\partial y} \right)_{y=0}, q_w'' = -\kappa_w \left(\frac{\partial T}{\partial y} \right)_{y=0} = -\frac{\kappa_w}{\rho_{nf} c_{p,nf}} \left(\frac{\partial \hat{h}}{\partial y} \right)_{y=0}, j_w'' = -D_{nf} \left(\frac{\partial C}{\partial y} \right)_{y=0} \quad (31)$$

thermal conductivity at the wall, Skin friction coefficient, Nusselt number and Sherwood number is defined as

$$\kappa_w = \kappa_\infty [1 + \gamma h(0)], C_f = \frac{\tau_w}{\frac{1}{2} \rho_f U_w^2}, Nu = \frac{x q_w''}{\kappa_\infty \hat{h}_w}, Sh = \frac{x j_w''}{D_{nf} (C_w - C_\infty)} \quad (32)$$

Applying Equations (12)–(14) to (31), the dimensionless engineering quantities becomes

$$Cf Re^{1/2} = 2 \left(\frac{\mu_{nf}}{\mu_f} \right) f''(0), Nu Re^{-1/2} = -[1 + \gamma h(0)] h'(0), Sh Re^{-1/2} = -\phi(0) \quad (33)$$

3. Results and Discussion

A coupled transport in MHD Casson nanofluid flow under volumetric sensible-enthalpy formulation, revealing energy pathways beyond wall fluxes was considered. The governing partial differential Equations (2)-(4) subject to the boundary conditions (5)-(6) were reduced using similarity variables (12)-(13) to a coupled nonlinear ordinary differential Equations (15)-(17) with dimensionless boundary conditions in (18)-(19). The system of first order ODE was solved numerically using collocation method implemented in the solve_bvp routine of the SciPy library. The influence of the thermo-physical parametric analyses on the velocity, enthalpy, and concentration distributions is summarized in tabular form and illustrated graphically. The selected parameter values are consistent with those commonly adopted in the literature, and the corresponding baseline parameters are as follows:

$\beta = 0.2, Gr_t = 5.0, Gr_s = 2.0, M = 3.0, Pr_{nf} = 143, Nb = 0.1, Sc = 0.62, Ec_h = 1.0, Br_h = 0.01,$

Table 1 displays the base thermophysical properties of Al_2O_3 –ethylene glycol used in this study. The thermal conductivity and viscosity of ethylene glycol and Al_2O_3 –ethylene glycol nanofluids at temperatures spanning 283–323 K have been experimentally reported by [12]. Specific heat capacity data for Al_2O_3 and ethylene glycol nanofluids are based on measurements [29]. These sources provide values that support the approximate base properties used here.

Table 1: Base physical properties of Al_2O_3 –ethylene glycol nanofluid at 25°C

Property	Ethylene Glycol (Base fluid)	Al_2O_3 (Nanoparticles)
Density (ρ)	1110.0	3970.0
Specific capacity (c_p)	2415.0	765.0
Thermal conductivity (κ)	0.25	36.0

Table 2: C_f with Nb for different values of M

M	Nb	C_f
1.0	0.05	-0.5103288205063
1.0	0.1	-0.5102419762794
1.0	0.2	-0.5100678381552
1.0	0.3	-0.5098930976441
2.0	0.05	-0.6753711977589
2.0	0.1	-0.6752103403988
2.0	0.2	-0.6748878111238
2.0	0.3	-0.6745641909003
3.0	0.05	-0.8251779213817
3.0	0.1	-0.8249521861829
3.0	0.2	-0.8244995938047
3.0	0.3	-0.8240454990106
5.0	0.05	-1.0920738091469
5.0	0.1	-1.0917386306259
5.0	0.2	-1.091066662697
5.0	0.3	-1.0903925383055

Figure 2 illustrates the variation of the skin-friction coefficient $C_f Re^{\frac{1}{2}}$ with the Brownian motion parameter Nb for different values of the magnetic parameter M in an Al_2O_3 –ethylene glycol nanofluid. The skin-friction coefficient shows negligible sensitivity to variations in Nb (from 0.05 to 0.30) across all M , indicating that Brownian diffusion primarily influences the internal thermal and concentration fields without directly affecting wall shear stress. In contrast, increasing M from 1 to 5 causes a pronounced reduction in the skin-friction coefficient (more negative values), reflecting a significant increase in flow resistance at the wall due to the Lorentz force. This clear decoupling highlights that magnetic effects (M) dominantly control the hydrodynamic wall response, while Brownian effects (Nb) are confined to thermo-solutal transport within the fluid volume. This separation, discernible through the volumetric enthalpy formulation, exemplifies the core advantage of the present approach: it distinguishes mechanisms controlling wall-transfer rates from those governing internal energy and species pathways, a distinction often obscured in conventional temperature-gradient analyses.

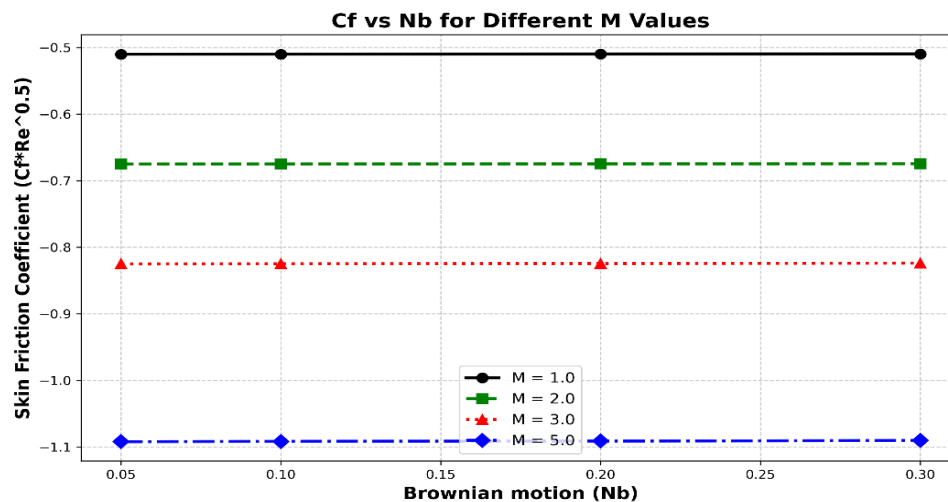


Figure 2. Variation of C_f with Nb for different M values

Table 3: Nu with Pr for different value of β

β	Pr_{nf}	Nu
0.1	100	-0.1576137920587
0.1	125	-0.1649765862556
0.1	150	-0.1710564442881
0.1	175	-0.176239613744
0.1	200	-0.1807598493767
0.2	100	-0.1459021576609
0.2	125	-0.1530585888260
0.2	150	-0.1589783298508
0.2	175	-0.1640317390021
0.2	200	-0.1684435251055
0.5	100	-0.1351651512912
0.5	125	-0.1421015583621
0.5	150	-0.147852391272
0.5	175	-0.1527678888243
0.5	200	-0.1570646243708
1.0	100	-0.130070522577

1.0	125	-0.1368863704084
1.0	150	-0.1425422671405
1.0	175	-0.1473822984193
1.0	200	-0.1516161506774

Figure 3 depicts the variation of the reduced Nusselt number $NuRe^{-1/2}$ with the nanofluid Prandtl number Pr_{nf} for different Casson parameter values β in the Al_2O_3 –ethylene glycol-based nanofluids. An increase in Pr_{nf} consistently reduces the Nusselt number for all β , indicating weakened wall heat transfer due to lower thermal diffusivity relative to momentum diffusivity. In the enthalpy-based formulation, higher Pr_{nf} suppresses internal energy diffusion across the thermal boundary layer, leading to a reduced wall enthalpy gradient and diminished conductive heat flux. For a fixed Pr_{nf} , increasing β significantly enhances the Nusselt number (Nu). Physically, larger β corresponds to reduced yield-stress effects, improving fluid mobility and momentum transport near the wall, which strengthens convective enthalpy transport and elevates heat transfer rates. These results highlight the competing roles of thermal diffusion governed by Pr_{nf} and rheological resistance controlled by β . The findings emphasize the importance of rheology–thermal coupling in Casson nanofluids and demonstrate that reducing yield-stress effects is crucial for maintaining efficient heat transfer in high Pr_{nf} regimes.

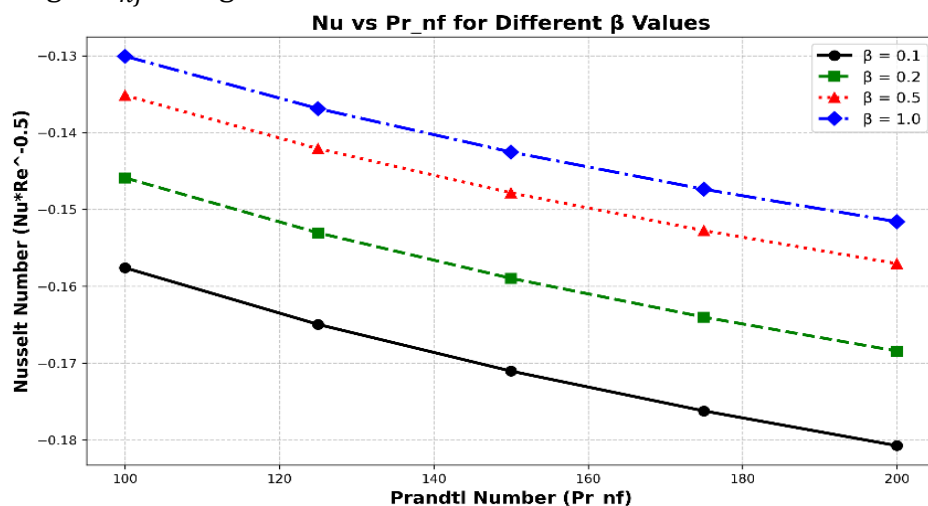


Figure 3. Variation of Nu with Pr_{nf} for different vales of β

Table 4: Effects of Sh and Sc for different vales of M

M	Sc	Sh
1.0	0.4	0.4340463927541
1.0	0.5	0.4897911837780
1.0	0.62	0.5504133669275
1.0	0.8	0.6320633351340
2.0	0.4	0.4145633061315
2.0	0.5	0.4696778295694
2.0	0.62	0.5297960100333
2.0	0.8	0.6109851220468
3.0	0.4	0.3976127324007
3.0	0.5	0.4521305896572
3.0	0.62	0.5117667537028

3.0	0.8	0.5925082009890
5.0	0.4	0.3692059036086
5.0	0.5	0.42250700098466
5.0	0.62	0.48113710391280
5.0	0.8	0.5609209644201

Figure 4 illustrates the variation of the reduced Sherwood number $Sh Re^{-1/2}$ with Schmidt number Sc for different magnetic parameter values of M in the present nanofluid. For all M , the Sherwood number increases monotonically with Sc , reflecting reduced mass diffusivity relative to momentum diffusivity and the consequent thinning of the concentration boundary layer. This leads to steeper wall concentration gradients and enhanced diffusive mass transfer. At a fixed Sc , increasing the magnetic parameter M markedly decreases the Sherwood number. The imposed transverse magnetic field induces a Lorentz force that retards fluid motion, weakens convective transport, and thickens the solutal boundary layer, thereby reducing the wall mass flux. This suppressive effect becomes more pronounced at higher Sc , where mass transfer is otherwise diffusion-dominated. Within the volumetric sensible-enthalpy framework, these trends demonstrate the competing influences of diffusion enhancement due to higher Sc and convective attenuation caused by magnetic damping. The results highlight the effectiveness of magnetic fields as a control mechanism for regulating nanoparticle mass transfer in nanofluid systems.

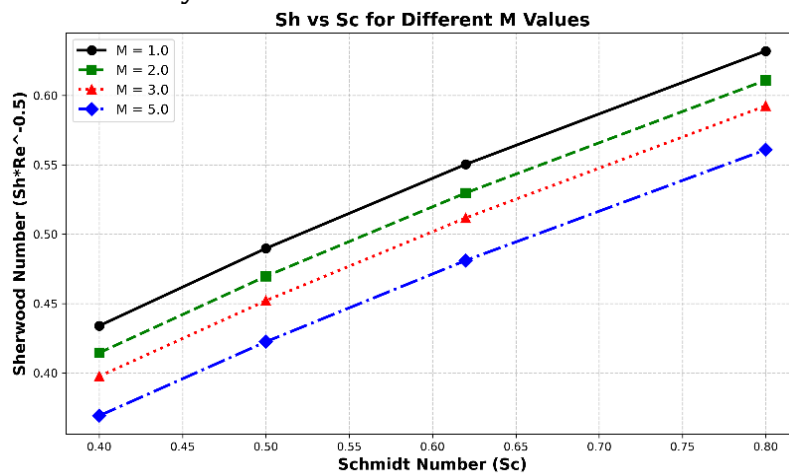


Figure 4. Variation of Sh and Sc for different values of M

Table 5: Numerical computational effects of C_f with Gr_t for different values of Gr_s

Gr_s	Gr_t	C_f
0.5	1.0	-1.5233462094851
0.5	3.0	-1.3826700984103
0.5	5.0	-1.234999526752
0.5	8.0	-0.9987092986578
0.5	10.0	-0.8300270155312
1.0	1.0	-1.3978139098906
1.0	3.0	-1.2496967197559
1.0	5.0	-1.0940878979074
1.0	8.0	-0.8449018411015
1.0	10.0	-0.6669312729140
2.0	1.0	-1.156676969882

2.0	3.0	-0.9949728397609
2.0	5.0	-0.8249521861829
2.0	8.0	-0.5525402389575
2.0	10.0	-0.3579863107303
5.0	1.0	-0.4861521204291
5.0	3.0	-0.2908907158815
5.0	5.0	-0.0856730411989
5.0	8.0	0.24262088297255
5.0	10.0	0.47646198323450

Figure 5 illustrates the variation of the scaled skin-friction coefficient $C_f Re^{1/2}$ with the thermal Grashof number Gr_t for various solutal Grashof numbers ($Gr_s = 0.5, 1.0, 2.0, 5.0$) for the considered fluid system. The coefficient increases monotonically with Gr_t for all Gr_s , reflecting enhanced fluid acceleration and wall shear stress due to stronger thermal buoyancy. For a fixed Gr_t , increasing Gr_s further augments $C_f Re^{1/2}$, demonstrating a synergistic effect between thermal and solutal buoyancy. At sufficiently high Gr_s , this combined buoyancy shifts the skin friction from negative to weakly positive values, indicating a change in the momentum boundary layer structure. Within the volumetric sensible-enthalpy framework, this trend results from the direct conversion of buoyancy-induced internal energy gradients into kinetic energy. The increase with both Gr_t and Gr_s shows how the formulation captures enhanced wall momentum transport through coupled thermal and solutal enthalpy. The synergy underscores the model's ability to unify dual buoyancy mechanisms thermal expansion and concentration-induced density variation into a single energy transport description, predicting combined flow acceleration and capturing regimes such as incipient flow reversal. These findings indicate that simultaneously increasing thermal and solutal buoyancy via greater temperature and concentration gradients can effectively enhance wall-driven momentum transport in buoyancy-assisted systems, with relevance for mixed-convection cooling and materials processing.

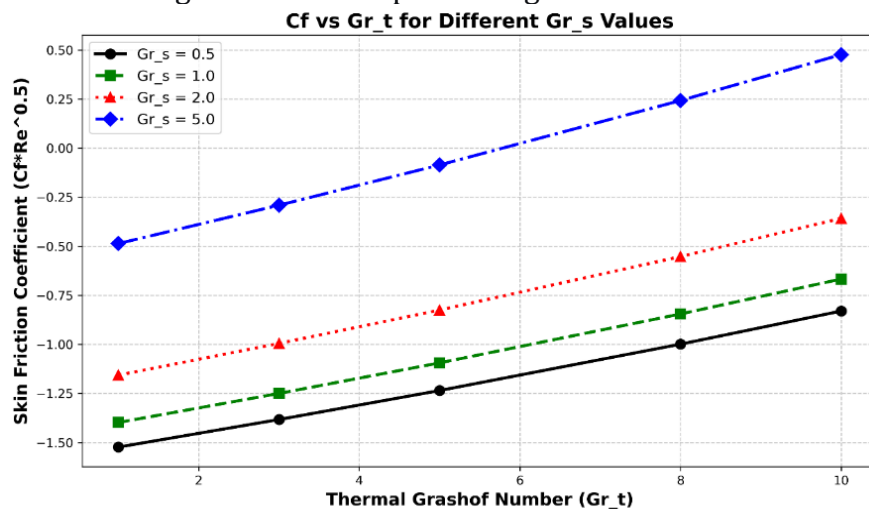


Figure 5. Variation of C_f and Gr_t for different values of Gr_s

Table 6: Tabulated values of Sh with Gr_s for different values of Gr_t

Gr_s	Gr_t	Sh
0.5	1.0	0.4682534056625
1.0	1.0	0.4799103498701

2.0	1.0	0.4997388521543
5.0	1.0	0.5440805187597
0.5	3.0	0.4737532637507
1.0	3.0	0.4855527928455
2.0	3.0	0.5056229805050
5.0	3.0	0.5504277545198
0.5	5.0	0.4795166524728
1.0	5.0	0.4914586405522
2.9	5.0	0.5117667537028
5.0	5.0	0.5570157913399
0.5	8.0	0.4887053651838
1.0	8.0	0.5008581046618
2.0	8.0	0.5215106909401
5.0	8.0	0.5673786784373
0.5	10.0	0.4952308514967
1.0	10.0	0.5075195473307
2.0	10.0	0.5283885365113
5.0	10.0	0.5746287000602

Figure 6 presents the variation of the reduced Sherwood number $Sh Re^{-1/2}$ with the solutal Grashof number Gr_s for different thermal Grashof numbers ($Gr_t = 1.0, 3.0, 5.0, 8.0$) for the considered fluid system. The Sherwood number increases monotonically with Gr_s for all Gr_t , indicating enhanced convective mass transfer due to stronger solutal buoyancy. For a fixed solutal Grashof number (Gr_s), increasing thermal Grashof number (Gr_t) further elevates the Sherwood number, demonstrating a synergistic interaction where thermal buoyancy augments the solutal-driven flow and intensifies the wall concentration gradient. Within the present model framework, this synergy arises from the coupled contribution of thermal and solutal effects to the total enthalpy, which drives convective transport. Increased thermal buoyancy converts more internal energy to kinetic energy, amplifying the velocity field responsible for species transport. This result highlights the model's capacity to predict coupled heat and mass transfer by unifying both buoyancy mechanisms in a single energy-based formulation. Practically, this synergy can be leveraged to optimize mass transfer in nanofluid applications, such as catalytic reactors or separation processes, through simultaneous control of thermal and solutal gradients.

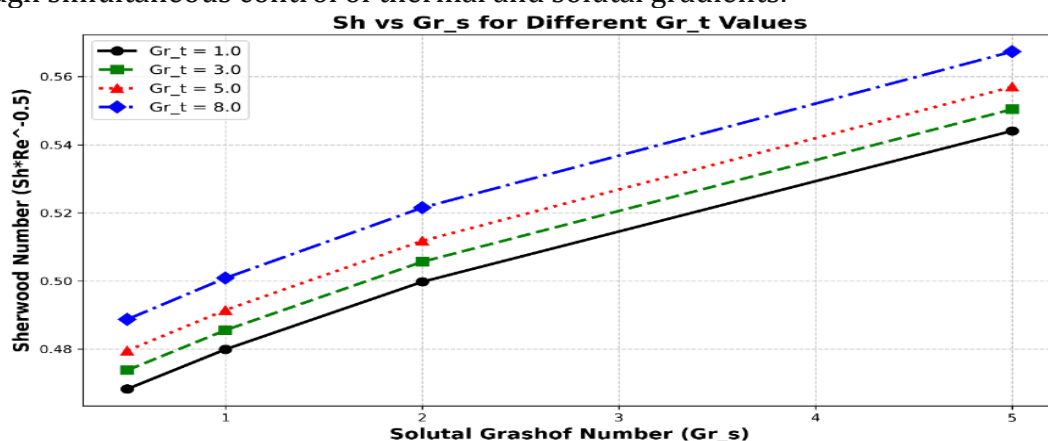


Figure 6. Variation of Sh with Gr_s for different Gr_t values

Table 7: Nu with Br_h for different Magnetic field values

M	Br_h	Nu
1.0	0.01	-0.0433656866383
1.0	0.05	-0.0433868570310
1.0	0.1	-0.0434133137013
1.0	0.5	-0.0436247144199
1.0	1.0	-0.0438883345829
2.0	0.01	-0.1017233269686
2.0	0.05	-0.1017562063612
2.0	0.1	-0.1017972962958
2.0	0.5	-0.1021256437599
2.9	1.0	-0.1025351491743
3.0	0.01	-0.1574205281773
3.0	0.05	-0.1574661661242
3.0	0.1	-0.1575232012783
3.0	0.5	-0.1579789915939
3.0	1.0	-0.1585475034886
5.0	0.01	-0.2639240529292
5.0	0.05	-0.2639976809173
5.0	0.1	-0.2640896977495
5.0	0.5	-0.2648251065934
5.0	1.0	-0.2657425545730

Figure 7 illustrates the variation of the reduced Nusselt number $Nu Re^{-1/2}$ with the Brinkman number Br_h for different magnetic parameter values ($M = 1.0, 2.0, 3.0, 5.0$). The results confirm that the Nusselt number exhibits negligible sensitivity to variations in Br_h across all magnetic field strengths, indicating that viscous dissipation contributes minimally to surface heat transfer within the examined parameter range. This behavior is explained within the volumetric sensible-enthalpy framework, where viscous heating primarily supplements the internal energy storage of the fluid rather than enhancing the wall-normal thermal gradient responsible for conductive heat flux. This clear partitioning—where viscous work contributes to volumetric energy storage rather than wall transfer—is precisely the kind of internal pathway revealed by the enthalpy formulation, moving beyond the assumption that all generated heat directly impacts surface flux. For a fixed Br_h , increasing the magnetic parameter M leads to a pronounced reduction in $Nu Re^{-1/2}$, reflecting the suppressive effect of the Lorentz force on fluid motion. Enhanced magnetic damping thickens the thermal boundary layer, weakens convective heat transport, and reduces the rate of heat transfer at the surface. The near-parallel nature of the curves further confirms that magnetic effects dominate over viscous dissipation in controlling surface heat transfer in the present MHD nanofluid system. From an engineering perspective, magnetic field regulation emerges as a more effective control strategy than viscous dissipation for tuning surface heat transfer in enthalpy-based MHD nanofluid thermal systems. This insight is particularly valuable for applications where active thermal management is required, such as in MHD-based heat exchangers or cooling devices, where adjusting the magnetic field strength offers precise control over heat transfer rates without relying on flow-induced viscous heating effects.

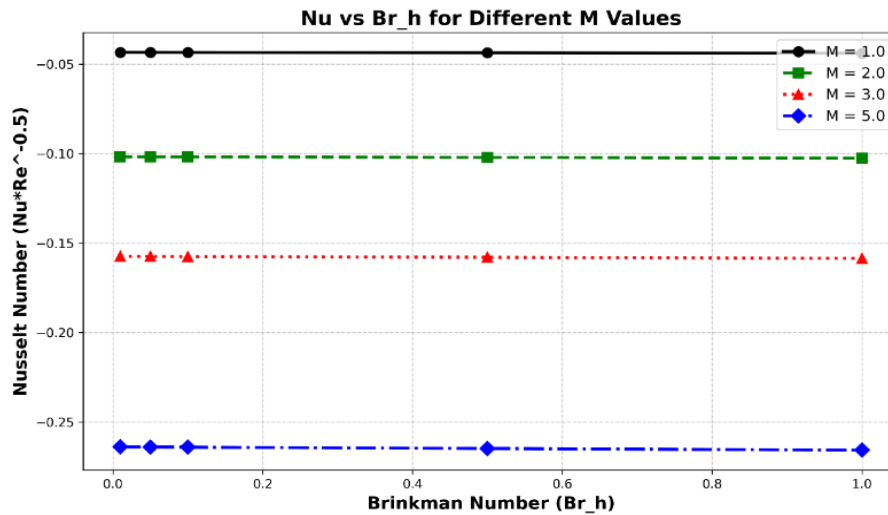


Figure 7. Variation of Nu vs Br_h for different M values

Table 8: Nu vs. Bi for Different Pr_{nf} values

Pr_{nf}	Bi	Nu
100	0.1	-0.0739633954530
100	0.2	-0.1459021576609
100	0.5	-0.3503951332875
100	1.0	-0.6577748358559
100	2.0	-1.1723645079990
125	0.1	-0.0774822357682
125	0.2	-0.1530585888260
125	0.5	-0.3690537056224
125	1.0	-0.6969747177599
125	2.0	-1.2547911422140
150	0.1	-0.0803951978620
150	0.2	-0.1589783298508
150	0.5	-0.3844689773184
150	1.0	-0.7293685537082
150	2.0	-1.3232083643823
175	0.1	-0.0828834056023
175	0.2	-0.1640317390021
175	0.5	-0.3976138827235
175	1.0	-0.7569890425817
175	2.0	-1.3817190286182

Figure 8 presents the variation of the reduced Nusselt number $Nu Re^{-1/2}$ with the Biot number Bi for different nanofluid Prandtl numbers ($Pr_{nf} = 100.0, 125.0, 150.0, 175.0$) of Al_2O_3 --ethylene glycol. The results demonstrate a clear monotonic decrease in $Nu Re^{-1/2}$ with increasing Bi across all Pr_{nf} values. This behavior indicates that enhanced surface convection (higher Bi) progressively reduces the wall heat transfer rate, as convective resistance at the boundary becomes more dominant relative to internal conductive heat transfer within the fluid. For a fixed Biot number, higher Pr_{nf} values consistently yield lower Nusselt numbers. Since $Pr_{nf} = \nu/\alpha_{nf}$ represents the ratio of momentum to thermal diffusivity, larger Pr_{nf}

values correspond to reduced thermal diffusion, which thickens the thermal boundary layer and suppresses conductive heat transfer from the wall. The parallel nature of the curves confirms that the influence of Bi is systematic across the examined range of thermal diffusivities, while the vertical separation between curves highlights the strong modulating effect of nanofluid thermal properties. Within the volumetric sensible-enthalpy framework, this interaction between Bi and Pr_{nf} reveals how boundary conditions and bulk fluid properties jointly govern heat transfer. The formulation captures the transition from conduction-dominated to convection-dominated regimes as Bi increases, while simultaneously accounting for the reduced thermal penetration depth associated with higher Pr_{nf} values. The engineering implication is significant: in nanofluid-based thermal systems, achieving optimal heat transfer requires careful balancing of surface convection conditions (characterized by Bi) and nanofluid thermal properties (characterized by Pr_{nf}). At high Bi and large Pr_{nf} , where heat transfer is most limited, design strategies should focus on either enhancing nanofluid thermal conductivity or adjusting surface convection characteristics to maintain adequate thermal performance.

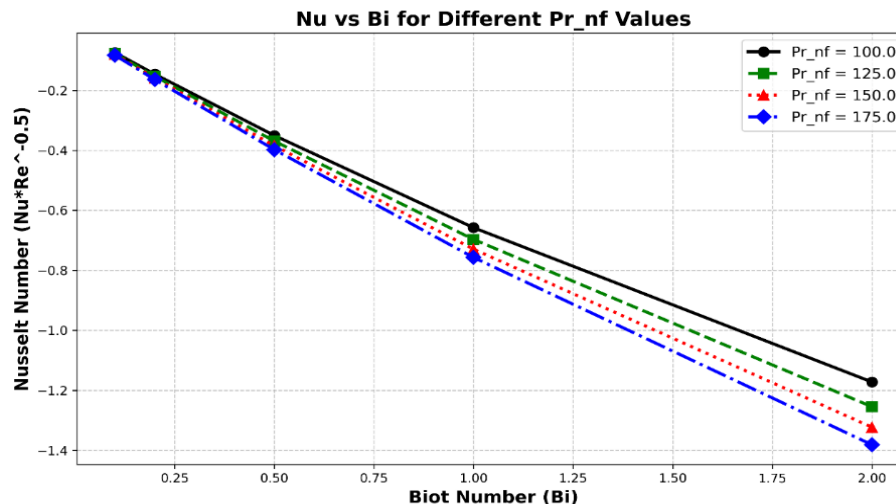


Figure 8. Variation of Nu with Bi for different Pr_{nf} values

Table 9. Nu and M for different Pr_{nf} values

M	Pr_{nf}	Nu
1.0	100	-0.0397529163974
2.0	100	-0.0942327167289
3.0	100	-0.1459021576609
5.0	100	-0.2439860596069
1.0	125	-0.0420011803138
2.0	125	-0.0988906846690
3.0	125	-0.1530585888260
5.0	125	-0.2563542341473
1.0	150	-0.0438519524315
2.0	150	-0.1027338119433
3.0	150	-0.1589783298508
5.0	150	-0.2666330934625
1.0	175	-0.0454256474777
2.0	175	-0.1060076874496

3.0	175	-0.1640317390030
5.0	175	-0.2754408845439
1.0	200	-0.0467949750203
2.0	200	-0.1088609110020
3.0	200	-0.1684435251055
5.0	200	-0.2831546405413

Figure 9 shows the reduced Nusselt number, $Nu Re^{-1/2}$, decreasing with the magnetic parameter M for different nanofluid Prandtl numbers ($Pr_{nf} = 100 - 175$) of Al_2O_3 --ethylene glycol. This indicates a consistent suppression of convective heat transfer by the magnetic field, attributable to the Lorentz force retarding flow and thickening the thermal boundary layer. For a fixed M , higher Pr_{nf} yields a lower Nusselt number. For the studied Al_2O_3 --ethylene glycol nanofluid, increased Pr_{nf} signifies a dominant rise in effective viscosity, which amplifies the Lorentz damping. The parallel curves confirm that magnetic suppression operates uniformly across this range. The results, framed within a volumetric sensible-enthalpy formulation, establish M as a key control parameter, where magnetic effects dominate thermal diffusion at higher values. This offers a direct mechanism for regulating heat transfer in MHD thermal systems. The greater suppression at higher Pr_{nf} further suggests that thermal performance can be tuned through nanoparticle loading. The distinct roles of magnetic damping (convective suppression) and Prandtl number (diffusive resistance) revealed here underscore the formulation's strength: it isolates how different physical mechanisms one external (Lorentz force) and one intrinsic (fluid properties)—separately modulate enthalpy transport from the volume to the wall.

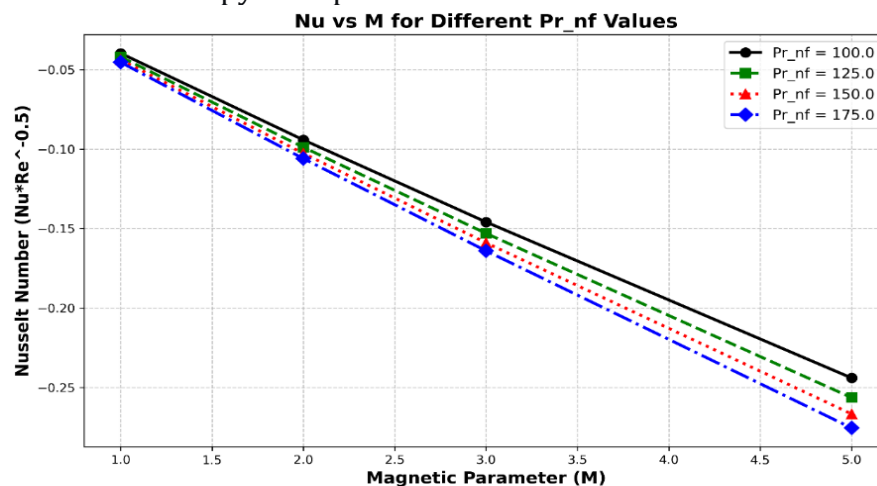


Figure 9. Variation of Nu with M for different values of Pr_{nf}

Table 10: Tabulated values of Sh with M for different values of Sc

Sc	M	Sh
0.4	1.0	0.434046393
0.4	2.0	0.414563306
0.4	3.0	0.397612732
0.4	5.0	0.369205904
0.5	1.0	0.489791184
0.5	2.0	0.46967783
0.5	3.0	0.45213059

0.5	5.0	0.422507001
0.62	1.0	0.550413367
0.62	2.0	0.52979601
0.62	3.0	0.511766754
0.62	5.0	0.481137104
0.8	1.0	0.632063335
0.8	2.0	0.610985122
0.8	3.0	0.592508201
0.8	5.0	0.560920964

Figure 10 presents the variation of the reduced Sherwood number $Sh Re^{-1/2}$ with the magnetic parameter M for different Schmidt numbers Sc in the present Al_2O_3 -ethylene glycol nanofluid. The Sc values are employed as dimensionless parameters to explore the momentum-to-mass-diffusivity ratio. A monotonic decrease in $Sh Re^{-1/2}$ with increasing M is observed for all Sc , indicating suppression of wall mass transfer by the magnetic field. This is attributed to the Lorentz force, which opposes fluid motion, weakens convective transport, and thickens the concentration boundary layer, thereby reducing the wall concentration gradient. For a fixed M , higher Schmidt numbers yield larger Sherwood numbers. Since $Sc = \nu/D$, an increase in Sc corresponds to lower mass diffusivity, which sharpens the near-wall concentration profile and enhances diffusive flux. This effect partially offsets the magnetic suppression of convection, revealing competition between magnetic damping and diffusive mass transfer. Within the volumetric sensible-enthalpy framework, these trends highlight the interplay between electromagnetic damping and species diffusion. The formulation captures how magnetic fields attenuate the convective component, while the Schmidt number governs diffusive resistance, a dual control mechanism crucial for systems requiring precise species transport regulation. From an engineering perspective, magnetic field strength serves as an active control parameter for modulating mass transfer in applications such as MHD-based separations, electrochemical reactors, and coating processes. Furthermore, by tailoring the effective Schmidt number through nanoparticle characteristics (size, concentration) or base-fluid properties, mass transfer performance can be fine-tuned under varying magnetic conditions. This quantified competition, resolvable within the volumetric enthalpy framework, exemplifies its utility: it disentangles the magnetic suppression of convection from the diffusive resistance governed by the Schmidt number, a coupling often conflated in models that consider only surface fluxes.

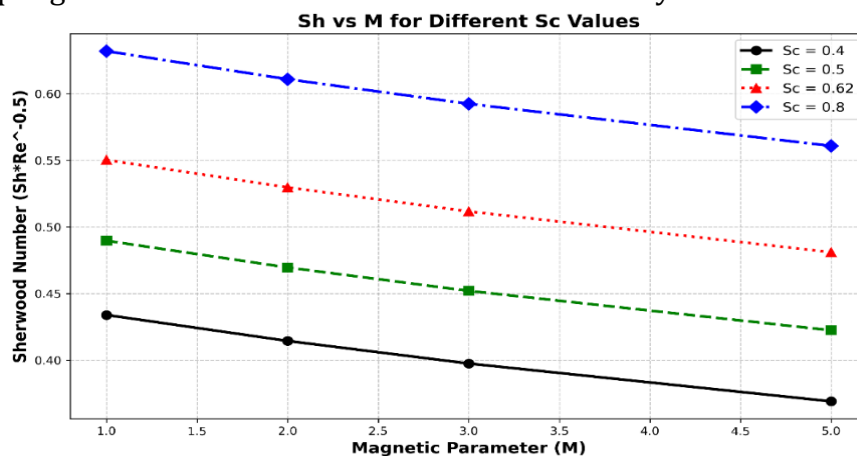


Figure 10. Variation of Sh with M for different Sc values

4. Conclusion

This study investigated magnetohydrodynamic Casson nanofluid flow and coupled heat-mass transfer over a stretching sheet using a volumetric sensible-enthalpy formulation. The effects of Brownian motion, magnetic field strength, Prandtl and Schmidt numbers, buoyancy parameters, viscous dissipation, Biot number, and rheological properties on skin friction, Nusselt number, and Sherwood number were systematically examined. The results revealed that:

1. The magnetic parameter exerts dominant control over momentum, heat, and mass transfer through Lorentz-force-induced damping.
2. Brownian motion has negligible influence on wall shear stress but plays a role in internal thermo-solutal transport.
3. Increasing Prandtl number suppresses wall heat transfer by reducing thermal diffusivity
4. Larger Casson parameter values enhance convective transport by weakening yield-stress resistance.
5. Schmidt number variations showed that reduced mass diffusivity sharpens concentration gradients and strengthens diffusive mass flux, partially offsetting magnetic suppression of species transport
6. Dual thermal and solutal buoyancy effects were found to synergistically accelerate flow and enhance wall momentum and mass transfer.
7. Viscous dissipation contributed minimally to surface heat transfer compared with magnetic damping.
8. Biot number governed surface convection resistance, while magnetic-Prandtl coupling demonstrated that Lorentz forces provide an effective means of regulating thermal transport in nanofluids.

These trends are consistent with classical MHD nanofluid studies in temperature-based formulations [30-35], although the present volumetric enthalpy approach provided additional insight into internal energy storage pathways, and successfully captured the interplay between electromagnetic damping, rheological behaviour, and thermo-solutal diffusion, providing a unified description of energy, momentum, and species transport in Casson nanofluid stretching-sheet systems. Similarly, this paper establishes that magnetic field strength is the most influential parameter controlling wall shear stress, heat transfer, and mass transfer in Casson nanofluid flow over a stretching sheet. Lorentz-force-induced damping thickens momentum, thermal, and concentration boundary layers, thereby suppressing convective transport at the surface. In contrast, Brownian motion primarily affects internal energy and concentration redistribution without significantly modifying wall momentum transport.

Thermal transport governed jointly by Prandtl number and Casson rheology: reduced thermal diffusivity weakens wall heat transfer, whereas diminished yield-stress effects enhance convective enthalpy transport. Mass transfer characteristics are dictated by Schmidt number and magnetic damping, with diffusion-driven enhancement competing against magnetic suppression of convection. Dual thermal and solutal buoyancy effects synergistically reinforce momentum and mass transport, confirming the importance of coupled buoyancy mechanisms in mixed-convection stretching-sheet flows. The volumetric sensible-enthalpy framework proved effective in unifying these coupled physical mechanisms, enabling accurate prediction of hydrodynamic, thermal, and species transport behaviours in complex MHD Casson nanofluid systems. The

results demonstrate that heat and mass transfer rates in Casson nanofluids can be actively regulated through magnetic field control and passively tuned through nanofluid thermophysical and rheological properties.

Our findings in this study are directly relevant to industrial and technological processes involving stretching surfaces where controlled heat and mass transfer are essential. Key application areas include:

1. Polymer extrusion and sheet stretching processes, where uniform cooling and surface quality depend on precise control of convective heat transfer.
2. Continuous casting and glass fiber drawing, where stretching-sheet flows govern solidification and material properties.
3. Coating and thin-film deposition technologies, where regulated mass transfer determines coating thickness and uniformity.
4. Textile and paper drying systems, where stretching surfaces and controlled heat fluxes improve energy efficiency.
5. MHD-assisted manufacturing systems, where magnetic fields are employed to regulate flow stability and thermal transport.
6. Micro-fabrication and roll-to-roll processing, where nanofluid-based cooling over moving sheets enhances thermal management.

The ability to manipulate momentum, heat, and mass transfer through magnetic fields and nanofluid design makes the present enthalpy-based Casson nanofluid stretching-sheet model valuable for next-generation materials processing and thermal management technologies.

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Nomenclature

h	Dimensionless enthalpy	$h = \frac{\hat{h} - \hat{h}_\infty}{\hat{h}_w - \hat{h}_\infty} = \frac{\hat{h}}{\hat{h}_w} = \frac{T - T_\infty}{T_w - T_\infty} = \theta$
$\hat{h}$	Dimensional volumetric sensible enthalpy	$\hat{h} = \rho_{nf} c_{p,nf} (T - T_\infty)$
$\hat{h}_w$	Wall stream volumetric sensible enthalpy	$\hat{h}_w = \rho_{nf} c_{p,nf} (T_w - T_\infty)$

$$\begin{array}{ll} \hat{h}_{\infty} & \text{Free stream volumetric sensible enthalpy } \hat{h}_{\infty} = 0 \\ T & \text{Local temperature field } T = T_{\infty} + \frac{\hat{h}}{\rho_{nf} c_{p,nf}} \end{array}$$

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